

THE CONJUGACY PROBLEM FOR THE AUTOMORPHISM GROUP OF THE RANDOM GRAPH

SAMUEL COSKEY, PAUL ELLIS, AND SCOTT SCHNEIDER

ABSTRACT. We prove that the conjugacy problem for the automorphism group of the random graph is Borel complete, and discuss the analogous problem for some other \aleph_0 -categorical structures.

1. INTRODUCTION

In [Tru1], Truss proved that the automorphism group of the random graph is simple. Truss continued to study the conjugacy relation on this group in [Tru2], proving that any element can be written as the product of at most three conjugates of any other. He did not, however, give a complete solution to the conjugacy problem for this group. In this paper we shall use the notion of Borel complexity to show that this problem is in fact as difficult as it could conceivably be.

In order to make this precise, we will use the theory of definable equivalence relations. A *standard Borel space* is a Polish space equipped just with its σ -algebra of Borel sets. For instance, \mathbb{R}^n , Cantor space $2^{\mathbb{N}}$, and the Baire space $\mathbb{N}^{\mathbb{N}}$ are all standard Borel spaces. Moreover, it is well-known that any Borel subset of a standard Borel space is again a standard Borel space in its own right. If X is a standard Borel space, an equivalence relation E on X is called *Borel* (analytic, *etc.*) iff it is Borel (analytic, *etc.*) as a subset of $X \times X$. It turns out that classification problems from many areas of mathematics may be realized as definable equivalence relations on suitably chosen standard Borel spaces.

For example, consider the problem of classifying all countable graphs up to graph isomorphism. Letting \mathcal{G} be the set of graphs of the form $\Gamma = \langle \mathbb{N}, R \rangle$ and identifying each graph $\Gamma \in \mathcal{G}$ with its edge relation $R \in 2^{\mathbb{N}^2}$, it is easily checked that \mathcal{G} is a Borel subset of $2^{\mathbb{N}^2}$ and hence is itself a standard Borel space. More generally, suppose that \mathcal{L} is a countable relational language and that σ is a sentence of the infinitary logic $\mathcal{L}_{\omega_1, \omega}$ in which countable conjunctions and disjunctions are allowed. Then

$$\text{Mod}(\sigma) = \{\mathcal{M} \mid \text{the universe of } \mathcal{M} \text{ is } \mathbb{N}, \text{ and } \mathcal{M} \models \sigma\}$$

is a standard Borel space, and the isomorphism relation \cong_σ on $\text{Mod}(\sigma)$ is an analytic equivalence relation (see for instance [HK]).

If E is an equivalence relation on the standard Borel space X , we say that E is *smooth*, or concretely *classifiable*, iff there exists a Borel function f from X into some standard Borel space Y such that for all $x, x' \in X$,

$$x E x' \iff f(x) = f(x').$$

In other words, f selects elements of Y as complete invariants for the classification problem for elements of X up to E . For instance, let $\mathcal{C}_{\mathbb{N}}$ denote the conjugacy relation on $\text{Aut } \mathbb{N}$ (*i.e.*, on the set of permutations $f : \mathbb{N} \rightarrow \mathbb{N}$). It is easy to see that a permutation of \mathbb{N} is completely determined up to conjugacy by its *cycle type*, that is, the number of n -cycles for $n = 1, 2, \dots, \infty$. Since the cycle type can be explicitly calculated and encoded as an element of the standard Borel space $\mathbb{N}^{\mathbb{N}}$, we have that $\mathcal{C}_{\mathbb{N}}$ is smooth.

If E is not smooth, it is still very useful to speak of its complexity relative to other classification problems. If E, F are equivalence relations on standard Borel spaces X, Y , then we say that E is *Borel reducible* to F , and write $E \leq_B F$, iff there exists a Borel map $f : X \rightarrow Y$ such that for all $x, x' \in X$,

$$x E x' \iff f(x) F f(x').$$

If $E \leq_B F$, then the classification problem associated to E should be regarded as no more difficult than that associated to F , in the sense that any set of complete invariants for F can be used for E as well. Now, \leq_B defines a partial preordering on all equivalence relations on standard Borel spaces, and a foundational goal of the subject is to understand the structure of this ordering.

It turns out that there is a \leq_B -maximum element among the classification problems \cong_σ introduced earlier. An equivalence relation E is said to be *Borel complete* iff for every countable relational language \mathcal{L} and every sentence σ of $\mathcal{L}_{\omega_1, \omega}$, we have $\cong_\sigma \leq_B E$. A classification problem which corresponds to a Borel complete equivalence relation should be regarded as totally intractable. It is not immediately clear that any Borel complete equivalence relations should exist, but it is shown in [FS] that the isomorphism relation for countable graphs is Borel complete, as is the isomorphism relation for countable groups.

Now, we may in fact consider a class of equivalence relations which is much broader than just the \cong_σ . An equivalence relation E is said to be *classifiable by countable structures* iff E is Borel reducible to \cong_σ for some $\mathcal{L}_{\omega_1, \omega}$ -sentence σ in a countable relational language \mathcal{L} .

Thus, a Borel complete equivalence relation is also universal for the class of equivalence relations which are classifiable by countable structures.

A large source of examples of equivalence relations which are classifiable by countable structures are those which arise from Polish group actions. If the Polish group G acts in a Borel fashion on the standard Borel space X , then the corresponding orbit equivalence relation E_G^X is an analytic equivalence relation. Here, E_G^X is defined by

$$x E_G^X x' \iff x \text{ and } x' \text{ lie in the same } G\text{-orbit.}$$

By [BK], the orbit equivalence relation E_G^X induced by a Polish group action is classifiable by countable structures. We remark that any \cong_σ is induced by the action of some Polish subgroup of S_∞ .

We shall now specialize to classification problems of the following form. If \mathcal{M} is any countable model, then $\text{Aut } \mathcal{M}$ denotes its automorphism group, and $\mathcal{C}_\mathcal{M}$ the conjugacy equivalence relation on $\text{Aut } \mathcal{M}$. That is,

$$f \mathcal{C}_\mathcal{M} g \iff (\exists h \in \text{Aut } \mathcal{M})(hf = gh)$$

Then $f, g \in \text{Aut } \Gamma$ are conjugate if and only if the expansions (Γ, f) and (Γ, g) are isomorphic, and so \mathcal{C}_Γ is classifiable by countable structures. (Alternatively, \mathcal{C}_Γ is the orbit equivalence relation induced by the conjugation action of the Polish group $\text{Aut } \Gamma$ on itself, and hence is classifiable by countable structures.) We are particularly interested in the complexity of $\mathcal{C}_\mathcal{M}$ for \mathcal{M} a model of some \aleph_0 -categorical theory, as there are several interesting examples in the recent literature.

Theorem ([For, Theorem 76]). *The conjugacy problem \mathcal{C}_Q for $\text{Aut}(Q, \leq)$ is Borel complete.*

Theorem ([CG, Theorem 5]). *Let \mathbb{B} denote the countable atomless boolean algebra. Then the conjugacy problem $\mathcal{C}_\mathbb{B}$ for $\text{Aut}(\mathbb{B})$ is Borel complete.*

We shall prove the analogous result for the automorphism group of the random graph. Recall that the *random graph* Γ is the unique countably infinite graph satisfying the homogeneity property: for any pair of finite and disjoint sets of vertices $U, V \subset \Gamma$, there exists a vertex x adjacent to each member of U and to no member of V .

Theorem. *The conjugacy problem \mathcal{C}_Γ for the automorphism group of the random graph is Borel complete.*

This gives the third of many conceivable examples of conjugacy problems for automorphism groups of \aleph_0 -categorical structures whose Borel complexity is the maximum

possible. However, in many other cases the conjugacy problem turns out to be smooth. For instance, we have already noted that the conjugacy problem for $\text{Aut } \mathbb{N}$ is smooth, and it is not hard to see that conjugacy problem for the automorphism group of the complete binary tree is smooth as well. This leaves open the question of which Borel complexities can arise as \mathcal{C}_M for some countable model M of an \aleph_0 -categorical theory.

Conjecture. *If M is the countable model in an \aleph_0 -categorical theory then \mathcal{C}_M is either smooth or Borel complete.*

There is an analogy between the proof of Foreman's theorem that \mathcal{C}_Q is Borel complete and our proof that \mathcal{C}_Γ is Borel complete. Foreman's method was to reduce the isomorphism problem for countable linear orders to \mathcal{C}_Q ; ours will be to reduce the isomorphism problem for countable graphs to \mathcal{C}_Γ . As a warm-up, we shall give a proof of Foreman's theorem in the next section. Then, in the final section, we give the proof of our theorem.

2. THE DLO

In this section, we shall record a detailed proof of the fact, due to Foreman [For, Theorem 76], that \mathcal{C}_Q is Borel complete. The argument is based on the following result.

2.1. Theorem ([FS, Theorem 3]). *The isomorphism relation for countable linear orders is Borel complete.*

We shall show:

2.2. Theorem. *The isomorphism relation for countable linear orders is Borel reducible to \mathcal{C}_Q .*

From this, we immediately obtain the desired conclusion.

2.3. Corollary. \mathcal{C}_Q on $\text{Aut } Q$ is Borel complete.

In what follows, an order-preserving map f from a linear order into Q shall be called *closed* iff the image of f is a closed subset of Q .

2.4. Lemma. *For any countable linear order x , there exists a closed order-preserving map from x into Q .*

Proof of Lemma 2.4. Start by using the standard back-and-forth argument to obtain an order-preserving map $\alpha : x \rightarrow Q$. If $\text{Im}(\alpha)$ is not closed in Q then there are “extra” limit points $B := \text{Im}(\alpha)' \setminus \text{Im}(\alpha)$ which we shall throw away. It is easy to see that $Q \setminus B$ must be dense in Q , and so there exists an isomorphism $i : Q \setminus B \rightarrow Q$. Now, the composition $i \circ \alpha$ is as desired. \square

Proof of Theorem 2.2. We must show that there is a Borel function $x \mapsto \phi_x$ from the set of countable linear orders on \mathbb{N} to $\text{Aut } \mathbb{Q}$ which satisfies:

$$x \text{ is isomorphic to } y \quad \text{iff} \quad \phi_x \text{ is conjugate to } \phi_y.$$

In fact, for each such x we shall define $\phi_x \in \text{Aut } \mathbb{Q}$ so that the fixed-point set of ϕ_x is isomorphic to x .

Given the countable linear order x , we first let $\alpha_x : x \rightarrow \mathbb{Q}$ be a closed embedding as in Lemma 2.4. We shall construct ϕ_x with fixed-point set equal to $\text{Im}(\alpha_x)$ as follows. First, we of course let ϕ_x be the identity function on $\text{Im}(\alpha_x)$. Next, since α_x is a closed embedding, it is easily seen that each $q \in \mathbb{Q} \setminus \text{Im}(\alpha_x)$ is contained in a unique interval $I \subset \mathbb{Q} \setminus \text{Im}(\alpha_x)$ whose endpoints are (extended) real numbers from either $\text{Im}(\alpha_x)$ or $\text{Im}(\alpha_x)'$ or $\pm\infty$. We wish to define ϕ_x on each such interval I in such a way that no element of I is fixed. This can easily be done by a back-and-forth construction: simply add to the order-preserving conditions of the inductive hypothesis the requirement that $\phi_x(q) > q$. This completes the construction of ϕ_x .

At the moment, our construction is far too vague to tell whether $x \mapsto \phi_x$ is a Borel assignment. However, our definition can be made canonical by fixing an enumeration of \mathbb{Q} in advance and using it to carry out all back-and-forth constructions.

Now, we must verify that $x \mapsto \phi_x$ is indeed a reduction from the isomorphism relation on linear orders to the conjugacy relation on $\text{Aut } \mathbb{Q}$. If $x \cong y$, then we can clearly line up the fixed points of ϕ_x with those of ϕ_y , but it is not immediately clear how to extend this function to conclude that ϕ_x is conjugate to ϕ_y . Following Glass [Gla], define the *orbitals* of an automorphism $\phi \in \text{Aut } \mathbb{Q}$ as the sets $\text{Conv}\{\phi^n(q) : n \in \mathbb{Z}\}$, for $q \in \mathbb{Q}$. Here, if $S \subset \mathbb{Q}$ then

$$\text{Conv } S = \{r : s < r < s' \text{ for some } s, s' \in S\}.$$

Notice that if \mathcal{O} is an orbital of ϕ , then either \mathcal{O} is a singleton fixed point, or $\phi(q) > q$ for all $q \in \mathcal{O}$, or else $\phi(q) < q$ for all $q \in \mathcal{O}$. Accordingly, we shall call the orbital \mathcal{O} “fixed,” an “up-bump,” or a “down-bump” for ϕ . If f is a function from the orbitals of ϕ into the orbitals of ϕ , then let us say that f is *parity-preserving* iff it sends fixed points to fixed points, up-bumps to up-bumps, and down-bumps to down-bumps.

Notice that we have arranged for the orbitals of ϕ_x to be either fixed points or else up-bumps between successive limit points of $\text{Im}(\alpha_x)$ (or $\{\pm\infty\}$). We shall use the following key fact.

2.5. Theorem ([Gla, Theorem 2.2.5]). *Let $\phi, \psi \in \text{Aut } \mathbb{Q}$ and suppose that there is an order- and parity-preserving bijection between the orbitals of ϕ and the orbitals of ψ . Then ϕ and ψ are conjugate in $\text{Aut } \mathbb{Q}$.*

Sketch of proof of Theorem 2.5. The main point is to show that any two up-bumps are conjugate. If $\mathcal{O} = \text{Conv}\{\phi^n(q)\}$ and $\mathcal{O}' = \text{Conv}\{\psi^n(q')\}$ are each up-bump orbitals, then we first define $\beta(\phi^n(q)) := \psi^n(q')$ for $n \in \mathbb{Z}$. Next, use a back-and-forth argument to define $\beta|_{(q, \phi(q))}$ to be any order-preserving bijection $(q, \phi(q)) \rightarrow (q', \psi(q'))$. Lastly, define $\beta(\phi^n(y)) := \psi^n(\beta(y))$ for any $y \in (q, \phi(q))$. Now, it is easy to check that this defines a bijection $\beta : \mathcal{O} \rightarrow \mathcal{O}'$ and that $\beta\phi = \psi\beta$. \square

To complete the proof of Theorem 2.2, suppose first that $x \cong y$. Then this isomorphism clearly induces an order-preserving bijection from the orbitals of ϕ_x onto the orbitals of ϕ_y . Hence, Theorem 2.5 implies that ϕ_x and ϕ_y are conjugate. Conversely, if there is $\beta \in \text{Aut } \mathbb{Q}$ such that $\beta\phi_x = \phi_y\beta$, then β maps the fixed points of ϕ_x to the fixed points of ϕ_y and hence induces an isomorphism between x and y . \square

3. THE RANDOM GRAPH

In this section, Γ always refers to a fixed copy of the random graph on the vertex set \mathbb{N} . Recall that $\text{Aut } \Gamma$ denotes the automorphism group of Γ and that \mathcal{C}_Γ denotes the orbit equivalence relation on $\text{Aut } \Gamma$ arising from the conjugation action of $\text{Aut } \Gamma$ on itself.

3.1. Theorem ([FS, Theorem 1]). *The isomorphism relation for countable graphs is Borel complete.*

We shall establish the following analog of Theorem 2.2.

3.2. Theorem. *The isomorphism relation for countable graphs is Borel reducible to \mathcal{C}_Γ .*

3.3. Corollary. *\mathcal{C}_Γ on $\text{Aut } \Gamma$ is Borel complete.*

Proof of Theorem 3.2. As in the proof of Theorem 2.2, we must show that there is a Borel function $x \mapsto \phi_x$ from the set of countable graphs on \mathbb{N} to $\text{Aut } \Gamma$ such that

$$x \text{ is isomorphic to } y \quad \text{iff} \quad \phi_x \text{ is conjugate to } \phi_y.$$

For each graph x with domain \mathbb{N} , we shall construct a graph Δ_x with the following properties:

- (a) Δ_x contains two isomorphic copies of x as induced subgraphs,
- (b) there exists an automorphism of Δ_x which interchanges the two copies of x , and

(c) Δ_x is isomorphic to the random graph Γ .

This will be done using the ideas of [MW, Lemma 2.1] and the comments that follow. Then, roughly speaking, we shall take ϕ_x to be the automorphism given in (b), thought of as an automorphism of Γ .

Take for the vertices of Δ_x the set $\mathbb{N} \times \mathbb{N}$, and let $\Delta_x(i) := \{i\} \times \mathbb{N}$, which we shall refer to visually as the i^{th} row of Δ_x . We shall put a copy of x in each of rows 0, 1, and leave the remaining rows blank. More precisely, let

$$(i, j) \sim_{\Delta_x} (i, k) \iff j \sim_x k \quad (i = 0, 1)$$

and if $i > 1$, then $(i, j) \sim_{\Delta_x} (i, k)$ for all $j, k \in \mathbb{N}$.

There are also edges going across the rows, defined as follows. First, for each $j \in \mathbb{N}$ place an edge joining $(0, j)$ with $(1, j)$. For each row $i > 1$, fix an enumeration $\langle S_n^i : n \in \mathbb{N} \rangle$ of the sets of i^2 vertices consisting of exactly i vertices from each of rows $0, \dots, i-1$. Then place an edge from each vertex (i, n) to each of the vertices in S_n^i . Thus, for each choice of i vertices from each of rows $0, \dots, i-1$, there is exactly one vertex in row i which is adjacent to these vertices and to no others in rows $0, \dots, i$.

It is easily seen that the map which interchanges any element of $\Delta_x(0)$ with its unique neighbor in $\Delta_x(1)$ is a partial automorphism of Δ_x . Furthermore, this map extends uniquely to an automorphism ϕ_x of Δ_x that preserves each row ≥ 2 setwise. Note that the graph Δ_x satisfies the homogeneity property of the countable random graph, and hence it is isomorphic to Γ . Using the back and forth argument, we can canonically choose an isomorphism of Δ_x with Γ . (Like the construction of Δ_x , the construction of this isomorphism may depend on the particular encoding of x .) Using this isomorphism, we may regard ϕ_x as an element of $\text{Aut } \Gamma$. This completes our construction.

It remains to show that $x \cong y$ iff ϕ_x and ϕ_y are conjugate elements of $\text{Aut } \Gamma$. Again using the fact that Δ_x and Δ_y are canonically isomorphic to Γ , this translates to showing that $x \cong y$ iff there exists a graph isomorphism $\alpha : \Delta_x \rightarrow \Delta_y$ such that $\alpha\phi_x = \phi_y\alpha$.

For the forward direction, if $\alpha : x \cong y$ then α induces a row-preserving isomorphism $\alpha : \Delta_x \cong \Delta_y$ which acts by α on the first two rows. Clearly, this α satisfies our requirements.

Conversely, let $\alpha : \Delta_x \rightarrow \Delta_y$ be a graph isomorphism satisfying $\alpha\phi_x = \phi_y\alpha$. We first show that if v lies in row 0 or 1 of Δ_x , then $\alpha(v)$ lies in row 0 or 1 of Δ_y . Indeed, if $\alpha(v)$ lies in some row ≥ 2 of Δ_y , then since ϕ_y preserves this row, we have that $\alpha(v)$ and $\phi_y\alpha(v)$ lie in the same row. In other words, $\alpha(v)$ and $\alpha\phi_x(v)$ lie in the same row of Δ_y . Hence, $\alpha(v)$ and $\alpha\phi_x(v)$ are not adjacent in Δ_y , contradicting that v and $\phi_x(v)$ were adjacent in Δ_x .

Similarly, α^{-1} maps rows 0, 1 of Δ_y to rows 0, 1 of Δ_x and it follows that α maps rows 0, 1 of Δ_x bijectively onto rows 0, 1 of Δ_y .

Now, let $n \in \mathbb{N}$ be a vertex of x , and consider the corresponding vertex $(0, n)$ of Δ_x . Then $\alpha((0, n))$ is in $\Delta_y(0)$ or $\Delta_y(1)$, and so it corresponds to a vertex $a(n)$ of y . We claim that a is an isomorphism between x and y . Indeed, the only way that this can fail is if some $\alpha((0, n)), \alpha((0, m))$ correspond to the same element of y , that is, $\alpha((0, m)) = \phi_y \alpha((0, n)) = \alpha \phi_x((0, n))$. However, this implies that $(0, m) = \phi_x((0, n)) = (1, n)$, a contradiction. \square

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